كلية الحاسبات و الذكاء الصناعي

المستوي: الثاني برامج (أمن المعلومات- شبكات)

الفصل الدراسي: الأول ٢٠٢١/٢٠٢٢

تاريخ الأختبار: ٢٠٢٢/١/١٥

نموذج اجابة : ورقة كاملة

اسم المقرر: Linear Algebra الجبر خطي

استاذ المقرر: د/مصطفى حسن عبدالله مطاوع

مدرس بقسم الرياضيات
كلية العلوم جامعة بنها

صورة من الأسئلة

Faculty of Computers & Artificial Intelligence,
Benha University

Student Name: ……………………………………………

Seat Number: ……………………………………………

Academic Year: 2021/2022

1st Term (January 2022) Final Exam

Program Name: Information Security

Course Name: Linear Algebra

Exam Date 15/01 / 2022

Question No	Marks	Examiner
Q1		
Q2		
Q3		
Q4		
Q5		
Q6		
Q7		
Q8		
Q9		
Q10		
Total		

Total Marks

50

Total Marks (in Letters)	…………………………………………………………………………………..		
Examination Committee	Examiner No. 1	Examiner No. 2	Examiner No. 3

Benha University
1^{st} Term (January 2022) Final Exam
Course Code: Level: 2^{nd} level
Subject: Linear Algebra

***Faculty of Computers* & Artificial Intelligence**
Date: 15 / 01 /2022
Time: 3 hrs.
Total Marks: 50 Marks
Examiner(s): Dr/ Mostafa Hassan

<u>Answer the following questions: (8 pages)</u>

Question No. 1 [13 Marks]

a) Solve the system by using back- substitution in row-echelon form

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

b) Is the set $W = \{(x, 1, z): x, z \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$ with the standard operations? Explain your answer.

Question No. 2 [13 Marks]

a) **Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by**

$$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$$

(1) Show that T is a Linear transformation

(2) Find Ker (T).

b) **Is the polynomial $x^3 + x$ belongs to span $\{x^3 - x, 3x^3 + 2x, x^2\}$. Explain your answer.**

Question No. 3 **[13 Marks]**

a) **Show that the set**

$$\boldsymbol{S} = \left\{ \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \right\}$$

is a basis of $\boldsymbol{M_{2,2}}$ with the standard operations.

b) **Find the conditions that must be achieved by scalar numbers $\boldsymbol{a}, \boldsymbol{b}$ to be the set of vectors**

$$\boldsymbol{S} = \{(\mathbf{1}, \mathbf{2}, \mathbf{0}), (\boldsymbol{a}, \boldsymbol{b}, \mathbf{2}), (\mathbf{1}, \mathbf{0}, \mathbf{1})\}$$

Linearly dependent.

Question No. 4 **[11 Marks]**

Find the eigenvalues and corresponding eigenvectors of the following matrix

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{5} & \mathbf{-2} & \mathbf{-4} \\ \mathbf{-2} & \mathbf{2} & \mathbf{2} \\ \mathbf{-4} & \mathbf{2} & \mathbf{5} \end{bmatrix}.$$

GOOD LUCK,

Dr. Mostafa Hassan

الأجـــابـة

Model Answer

[redacted]

Student Name: ....

Seat Number: ....

Faculty of Computers & Artificial Intelligence,
Benha University

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Total Marks (in Letters)	....		
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Answer the following questions: (8 pages)

Question No. 1 [13 Marks]

a) **Solve the system by using back- substitution in row-echelon form**

$$x - 2y + 3z = 9$$
$$-x + 3y = -4$$
$$2x - 5y + 5z = 17$$

b) **Is the set $W = \{(x, 1, z): x, z \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$ with the standard operations? Explain your answer.**

a) $\begin{pmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{pmatrix} \xrightarrow[-2R_1+R_3]{R_1+R_2} \begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$

$\begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \end{pmatrix} \xrightarrow{-R_2+R_3} \begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

$x - 2y + 3z = 9 \Rightarrow x = 2y - 3z + 9 = -2 - 6 + 9 = 1$
$y + z = 1 \Rightarrow y = 1 - z = -1$
$z = 2$
$\therefore x = 1, y = -1, z = 2$

b) Let $w_1 = (x_1, 1, z_1) \in W$, $w_2 = (x_2, 1, z_2) \in W \Rightarrow w_1 + w_2 = (x_1 + x_2, 2, z_1 + z_2) \notin W$
($x_1+x_2 \in \mathbb{R}$, $2 \neq 1$, $z_1+z_2 \in \mathbb{R}$)

$\therefore W$ is not a subspace of $\mathbb{R}^3$.

Question No. 2 [13 Marks]

a) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$
(1) Show that T is a Linear transformation

(2) Find Ker (T).

b) Is the polynomial $x^3 + x$ belongs to span $\{x^3 - x, 3x^3 + 2x, x^2\}$.
Explain your answer.

a) (1) Let $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$

$\Rightarrow X + Y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$, $\lambda X = (\lambda x_1, \lambda x_2, \lambda x_3)$

$\Rightarrow T(X+Y) = (x_1 + y_1 - x_2 - y_2 + x_3 + y_3,$
$2x_1 + 2y_1 + x_2 + y_2 - x_3 - y_3,$
$-x_1 - y_1 - 2x_2 - 2y_2 + 2x_3 + 2y_3)$

$= (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$
$+ (y_1 - y_2 + y_3, 2y_1 + y_2 - y_3, -y_1 - 2y_2 + 2y_3)$

$= T(X) + T(Y). \longrightarrow (1)$

$T(\lambda X) = T(\lambda x_1, \lambda x_2, \lambda x_3)$
$= (\lambda x_1 - \lambda x_2 + \lambda x_3, 2\lambda x_1 + \lambda x_2 - \lambda x_3, -\lambda x_1 - 2\lambda x_2 + 2\lambda x_3)$
$= (\lambda(x_1 - x_2 + x_3), \lambda(2x_1 + x_2 - x_3), \lambda(-x_1 - 2x_2 + 2x_3))$
$= \lambda(x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$
$= \lambda T(X) \longrightarrow (2)$

from (1) and (2) T is L.T.

2) $\text{Ker}(T) = \{(x_1, x_2, x_3) : T(x_1, x_2, x_3) = 0\}$

$T(x_1, x_2, x_3) = 0 \Rightarrow (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$
$= (0, 0, 0)$

$\Rightarrow$
$x_1 - x_2 + x_3 = 0$
$2x_1 + x_2 - x_3 = 0$
$-x_1 - 2x_2 + 2x_3 = 0$

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{pmatrix} \xrightarrow[R_1+R_3]{-2R_1+R_2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2+R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 - x_3 = 0 \\ x_3 = \alpha \end{array} \Rightarrow x_2 = \alpha, \ x_1 = 0$$

$$\therefore X \in Ker(T) : \begin{pmatrix} 0 \\ \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore Ker(T) = \left\{ \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R} \right\}$$

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b) $X^3 + X = C_1(X^3 - X) + C_2(3X^3 + 2X) + C_3 X^2$

$X^3 + X = (C_1 + 3C_2)X^3 + C_3 X^2 + (-C_1 + 2C_2)X$

$$\therefore \begin{array}{l} C_1 + 3C_2 = 1 \\ C_3 = 0 \\ -C_1 + 2C_2 = 1 \end{array} \Rightarrow \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_1+R_3}$$

$$\begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 5 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2/5 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow C_1 + 3C_2 = 1, \quad C_2 = 2/5, \quad C_3 = 0$

$\Rightarrow C_1 = 1 - 3C_2 = 1 - 6/5 = -1/5$

$\therefore C_1 = -\frac{1}{5}, \quad C_2 = \frac{2}{5}, \quad C_3 = 0$

$\Rightarrow X^3 + X = -\frac{1}{5}(X^3 - X) + \frac{2}{5}(3X^3 + 2X)$

$\Rightarrow X^3 + X \in Span\{X^3 - X, 3X^3 + 2X, X^2\}.$

Question No. 3 [13 Marks]

a) Show that the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of $M_{2,2}$ with the standard operations.

b) Find the conditions that must be achieved by scalar numbers a, b to be the set of vectors

$$S = \{(1,2,0), (a,b,2), (1,0,1)\}$$

Linearly dependent.

a) 1) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are Linearly independent

2) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = aV_1 + bV_2 + cV_3 + dV_4$

where $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $V_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow M_{2,2} = \text{Span}(S)$.

b) $C_1(1,2,0) + C_2(a,b,2) + C_3(1,0,1) = (0,0,0)$

$$\Rightarrow \begin{cases} C_1 + aC_2 + C_3 = 0 \\ 2C_1 + bC_2 = 0 \\ 2C_2 + C_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & a & 1 \\ 2 & b & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$A \quad C = 0$

S is Linearly dependent if $|A| = 0$

$$|A| = \begin{vmatrix} 1 & a & 1 \\ 2 & b & 0 \\ 0 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} b & 0 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} a & 1 \\ 2 & 1 \end{vmatrix} = b - 2(a-2) = b - 2a + 4$$

$\Rightarrow$ if $|A| = 0 \Rightarrow 2a - b = 4$. S is Linearly dependent If $2a - b = 4$.

Question No. 4 **[11 Marks]**

Find the eigenvalues and corresponding eigenvectors of the following matrix

$$A = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 2 & 2 \\ -4 & 2 & 5 \end{bmatrix}.$$

The ch. eq is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & -2 & -4 \\ -2 & 2-\lambda & 2 \\ -4 & 2 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)^2(\lambda-10)=0 \Rightarrow \lambda_1=\lambda_2=1, \lambda_3=10$$

at $\lambda_1 = 1$ we get

$$(A-I)X=0 \Rightarrow (A-I) = \begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{pmatrix}$$

$$\xrightarrow[-\frac{1}{4}R_3]{\frac{1}{4}R_1, -\frac{1}{2}R_2} \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & -1/2 & -1 \\ 1 & -1/2 & -1 \end{pmatrix} \xrightarrow[R_1-R_3]{R_1-R_2} \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \text{Rank}(A-I) = 1$$

$x_1 - \frac{1}{2}x_2 - x_3 = 0$, put $x_2 = \alpha$, $x_3 = \beta$

$$\Rightarrow x_1 = \alpha/2 + \beta$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha/2+\beta \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \\ \beta \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

at $\lambda=10$

$$(A-10I) = \begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix} \xrightarrow[\leftrightarrow R_1]{-\frac{1}{2}R_2} \begin{pmatrix} 1 & 4 & -1 \\ -5 & -2 & -4 \\ -4 & 2 & -5 \end{pmatrix}$$

$$\xrightarrow[4R_1+R_3]{5R_1+R_2} \begin{pmatrix} 1 & 4 & -1 \\ 0 & 18 & -9 \\ 0 & 18 & -9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \text{Rank}(A-10I)=2$

Put $x_3=\alpha \Rightarrow x_1+4x_2-x_3=0$

$2x_2-x_3=0$

$\Longrightarrow x_1=-4x_2+x_3=-2\alpha+\alpha=-\alpha$

$x_2=\frac{1}{2}\alpha$

$x_3=\alpha$

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -\alpha \\ \frac{1}{2}\alpha \\ \alpha \end{bmatrix}=\alpha\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

GOOD LUCK,

Dr. Mostafa Hassan