Answer Model

جامعة بنها – كلية الحاسبات

لطلاب المستوى الثانى

يوم الامتحان: الخميس ١٤ / ١ / ٢٠١٦ م

المادة : رياضيات (٣)

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اسئله + نموذج إجابه

ورقة كاملة

Benha University
Faculty of Computers and Informatics
Date: ٢٠١٦/١/١٤

Answer Model

Level: 2
Subject : Math. 3

Answer the following questions (marks ١٢٠)

Question 1.

1) **Show** that for two sets A and B

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

2) **Prove that** the set $F = \{1+x, 1-x, x^2, 3x^3\}$ is a base for $P_3(R)$.

3) Let R be the relation on $N \times N$ defined by $(a,b)R(c,d) \Leftrightarrow a+d=b+c$
   a) **Show that** R is an equivalence relation
   b) **Find** $[(1,1)]$ and $[(2,5)]$.

Question 2.

1) **Prove that** if $T: V \longrightarrow W$ and $S: W \longrightarrow Z$ are linear transformations, then the composition $S \circ T$ is also a linear transformation.

2) Let $f: R^2 \longrightarrow R^2$ be the function defined by $f(x,y) = (2x-3y, x-2y)$
   a) **Show that** f is a bijective function and **find** the inverse f^{-1}
   b) **Prove that** f is a linear transformation and **find** $Ker(f), Nullity(f)$.

Question 3.

1. Given $A = \begin{pmatrix} -2 & -9 \\ 1 & 4 \end{pmatrix}$, **Prove that**, for every positive integer n,

$$A^n = \begin{pmatrix} 1-3n & -9n \\ n & 1+3n \end{pmatrix}$$

2. If a connected planar graph G has V vertices and E edges and dividing the plane into F faces, then **prove that**: $F = E - V + 2$.

3. **Define** a Boolean algebra $(B, \oplus, *, \bar{\ }, 0, 1)$ and for all $b_1, b_2 \in B$, **prove that**:
   There is only one element $\overline{b_1} \in B$ such that $b_1 \oplus \overline{b_1} = 1$ and $b_1 * \overline{b_1} = 0$.

4. **Find** the inverse (if it exists) of the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 5 \end{pmatrix}$$

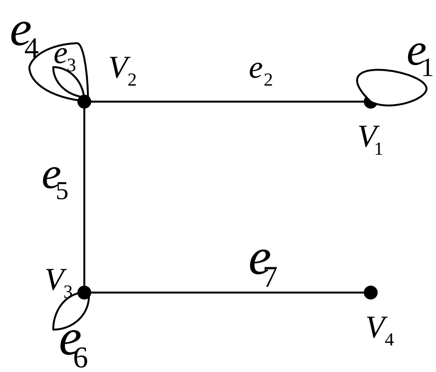
Question 2.

1. **Find** the value λ which make the following system has an infinite number of solutions and **write** the solution set of the system:

$$5x + 2y - z = 1$$
$$2x + 3y + 4z = 7$$
$$4x - 5y + \lambda z = \lambda - 5$$

2. **Define** *Trees, the complete graph* K_n , *the complete bipartite graph* $K_{r,s}$ and *Eulerian path*, and for **which values** of *n, r, s* , the graphs K_n , $K_{r,s}$ are **Eulerian**?

3. **Find** the matrix A^2 , where A be the adjacency matrix, for the following graph: and **write** all edge sequences of length 2 joining v_1, v_2 .



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Good Luck !

First Question:

1) *Show that for two sets A and B*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Sol.

We know that for two statements p, q *and* s, *we have*

$$p \vee (q \wedge s) \equiv (p \vee q) \wedge (p \vee s) \qquad (\Delta)$$

$$\begin{aligned} A \cup (B \cap C) &\stackrel{\text{def}}{=} \{x | x \in A \ \vee x \in (B \cap C)\} \\ &\stackrel{\text{def}}{=} \{x | x \in A \ \vee (x \in B \wedge x \in C)\} \\ &= \{x | x \in A \ \vee (x \in B \wedge x \in C)\} \\ &\triangleq \{x | (x \in A \ \vee x \in B) \wedge (x \in A \ \vee x \in C)\} \\ &= \{x | x \in (A \cup B) \wedge x \in (A \cup C)\} \\ &\stackrel{\text{def}}{=} (A \cup B) \cap (A \cup C) \end{aligned}$$

This completes the proof.

2) **Prove that** the set $F = \{1+x, 1-x, x^2, 3x^3\}$ is a base for $P_3(R)$.

Sol

$$a(1+x) + b(1-x) + c\,x^2 + d(3x^3) = 0$$

$$(a+b) + (a-b)x + c\,x^2 + 3dx^3 = A + Bx + Cx^2 + Dx^3$$

$$\frac{(A+B)}{2}(1+x) + \frac{(A-B)}{2}(1-x) + C\,x^2 + \frac{D}{3}(3x^3) = A + Bx + Cx^2 + Dx^3$$

$\{1+x, 1-x, x^2, 3x^3\}$ is span and linear independent so it is a base for $P_3(R)$

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3) *Let* R *be the relation on* $N \times N$ *defined by* $(a,b)R(c,d) \Leftrightarrow a+d = b+c$

c) *Show that* R *is an equivalence relation*

d) *Find* $[(1,1)]$ *and* $[(2,5)]$

Sol.

(a) *It is clear that* R *satisfies the following conditions*

(I) *R is reflexive :*

$$a + b = b + a \quad \forall\, a, b \in N \Rightarrow (a, b)R(a, b) \quad \forall\,(a, b) \in N \times N$$
$$\Rightarrow R \text{ is reflexive}$$

(II) *R is symmetric :*

$$(a, b)R\,(c, d) \Rightarrow a + d = b + c$$
$$\Rightarrow b + c = a + d$$
$$\Rightarrow c + d = d + a$$
$$\Rightarrow (c, d)R(a, b)$$

From (I) and (II), we conclude that R *is an equivalence relation on* $N \times N$

(b)

$$[(1,1)] \coloneqq \{(x, y) \in N \times N | (1,1)R(x, y)\}$$
$$= \{(x, y) \in N \times N | 1 + y = 1 + x\}$$
$$= \{(x, y) \in N \times N | y = x\}$$
$$= \{(x, x) | x \in N\}$$

$$[(2,5)] \coloneqq \{(x, y) \in N \times N | (2,5)R(x, y)\}$$
$$= \{(x, y) \in N \times N | 2 + y = 5 + x\}$$
$$= \{(x, y) \in N \times N | y = 3 + x\}$$
$$= \{(x, 3 + x) | x \in N\}$$

Second Question:

1) *Prove that if* $T: V \longrightarrow W$ *and* $S: W \longrightarrow Z$ *are linear transformations, then the composition* $S \circ T$ *is also a linear transformation.*

Sol:

Let $T: V \longrightarrow W$ *and* $S: W \longrightarrow Z$ *be two linear transformations, then*

$$(S \circ T)(u+v,z) = S(T(u+v,z))$$
$$= S(T(u,z)+T(v+z))$$
$$= S(T(u,z)) + S(T(v,z))$$
$$= (S \circ T)(u,z) + (S \circ T)(v,z)) \qquad (I)$$
$$(S \circ T)(\propto u,v) = S(T(\propto u,v))$$
$$= S(\propto T(u,v))$$
$$=\propto S(T(u,v))$$
$$=\propto (S \circ T)(u,v) \qquad (II)$$

Hence, from (I) and (II), we conclude that $S \circ T$ *is also a linear transformation.*

2) *Let* $f: R^2 \longrightarrow R^2$ *be the function defined by* $f(x,y) = (2x-3y, x-2y)$

c) *Show that* f *is a bijective function and find the inverse* f^{-1}

d) *Prove that* f *is a linear transformation and find* $Ker(f), Nullity(f)$.

Sol:

(a) *One can show that the function* f *satisfies the following conditions:*

(I) *f is injective:*

$$f(x,y) = f(\acute{x},\acute{y}) \Rightarrow (2x-3y, x-2y) = (2\acute{x}-3\acute{y}, \acute{x}-2\acute{y})$$
$$\Rightarrow 2x-3y = 2\acute{x}-3\acute{y} \;\wedge\; x-2y = \acute{x}-2\acute{y}$$
$$\Rightarrow x = \acute{x} \;\wedge\; y = \acute{y}$$
$$\Rightarrow (x,y) = (\acute{x},\acute{y})$$

Hence, f is injective.

(II) *f is surjective:*

Let $(u,v) \in R^2$ *such that* $f(x,y) = (u,v)$, *then*

$$\Rightarrow (u,v) = (2x-3y, x-2y)$$
$$\Rightarrow u = 2x-3y \;\wedge\; v = x-2y$$
$$\Rightarrow x = 2u-3v \;\wedge\; y = u-2v$$
$$\Rightarrow (x,y) = (2u-3v, u-2v) \qquad (*)$$

From (I) and (II), we conclude that f is bijective .

From (*), we obtain

$$f^{-1}(u,v) = (2u-3v, u-2v)$$

(b)

$$Ker(f) := \{(x,y) \in R^2 : f(x,y) = (0,0)\}$$
$$= \{(x,y) \in R^2 : (2x-3y, x-2y) = (0,0)\}$$
$$= \{(x,y) \in R^2 : 2x-3y = 0 \ \wedge x-2y = 0\}$$
$$= \{(x,y) \in R^2 : x = y = 0\}$$
$$= \{(0,0)\}$$

Hence, $Nullity(f) = 0$

Third Question :

1. Given $A = \begin{pmatrix} -2 & -9 \\ 1 & 4 \end{pmatrix}$, **Prove** that, for every positive integer n,

$$A^n = \begin{pmatrix} 1-3n & -9n \\ n & 1+3n \end{pmatrix}$$

Sol:

We begin $n=1$ i.e. $A^1 = \begin{pmatrix} 1-3.1 & -9.1 \\ 1 & 1+3.1 \end{pmatrix} = \begin{pmatrix} -2 & -9 \\ 1 & 4 \end{pmatrix} = A$

For $n=k$ let

Multiple by matrix A $A^k = \begin{pmatrix} 1-3k & -9k \\ k & 1+3k \end{pmatrix}$

$$A^{k+1} = \begin{pmatrix} 1-3k & -9k \\ k & 1+3k \end{pmatrix} . \begin{pmatrix} -2 & -9 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -2-3k & -9-9k \\ k+1 & 3k+4 \end{pmatrix} = \begin{pmatrix} 1-3(k+1) & -9(k+1 \\ k+1 & 1+3(k+1) \end{pmatrix}$$

We proved at $n=k+1$, completes the proof.

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2. If a connected planar graph G has V vertices and E edges and dividing the plane into F faces, then **prove that**: $F = E - V + 2$.

Sol:

The proof is by induction on the number of edges of G.
If $E = 0$ then $V = 1$ (G is connected, so there cannot be two or more vertices) and there is a single face (consisting of the whole plane except the single vertex), so $F = 1$.
therefore holds in this case.

Suppose, now, that the theorem holds for all graphs with fewer than n edges. Let G be a connected planar graph with n edges; that is $|E| = n$. If G is a tree, then $|V| = n + 1$ and $|F| = 1$, so the theorem holds in this case too.
If G is not a tree choose any cycle in G and remove one of its edges. The resulting graph G is connected, planar and has $n - 1$ edges, $|V|$ vertices and $|F| - 1$ faces.
By the inductive hypothesis, Euler's formula holds for G:
$|F| - 1 = (|E| - 1) - |V| + 2$
so $|F| = |E| - |V| + 2$
as required.

3. Define a Boolean algebra $(B, \oplus, *, \bar{\ }, 0, 1)$ and for all $b_1, b_2 \in B$, **prove that:**
There is only one element $\overline{b_1} \in B$ such that $b_1 \oplus \overline{b_1} = 1$ and $b_1 * \overline{b_1} = 0$.

Sol

Boolean algebra consists of a set B together with three operations defined on that set. These are:
(a) a binary operation denoted by $\oplus$ referred to as the **sum** ;
(b) a binary operation denoted by $*$ referred to as the **product** ;
(c) an operation which acts on a single element of B, denoted by -,
where, for any element $b \in B$, the element $b^- \in B$ is called the

complement of b^- (An operation which acts on a single member of a set S and which results in a member of S is called a **unary operation**.)

The following axioms apply to the set B together with the operations $\oplus$, $*$ and - .
B1. Distinct identity elements belonging to B exist for each of the binary operations $\oplus$ and $*$ and we denote these by **0** and **1** respectively. Thus we have

$$b \oplus \mathbf{0} = \mathbf{0} \oplus b = b$$
$$b * \mathbf{1} = \mathbf{1} * b = b \quad \text{for all } b \in B.$$

for all $a, b, c \in B$.

$$(a * b) * c = a * (b * c)$$
$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

B2. The operations $\oplus$ and $*$ are associative, that is
B3. The operations $\oplus$ and $*$ are commutative, that is

$$a \oplus b = b \oplus a$$
$$a * b = b * a \quad \text{for all } a, b \in B.$$

B4. The operation $\oplus$ is distributive over $*$ and the operation $*$ is distributive over $\oplus$, that is

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$
$$a * (b \oplus c) = (a * b) \oplus (a * c) \text{ for all } a, b, c \in B.$$

B5. For all b ∈ B, b ⊕ $\bar{b}$ = 1 and b ∗ $\bar{b}$ = 0.

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There is only one element $\overline{b_1} \in B$ such that $b_1 \oplus \overline{b_1} = 1$ and $b_1 * \overline{b_1} = 0$.

$$b \oplus \overline{b_1} = b_1 \oplus b = 1 \quad b \oplus \overline{b_2} = b_2 \oplus b = 1$$

$$b * \overline{b_1} = b_1 * b = 0, \quad b * \overline{b_2} = b_2 * b = 0$$

Thus we have

$\overline{b_1} = \overline{b_1} * 1$ (axiom B1)

$= \overline{b_1} * (b \oplus \overline{b_2})$

$= (\overline{b_1} * b) \oplus (\overline{b_1} * \overline{b_2})$ (axiom B4)

$= 0 \oplus (\overline{b_1} * \overline{b_2})$

$= 0 \oplus (\overline{b_2} * \overline{b_1})$ (axiom B3)

$= (\overline{b_2} * b) \oplus (\overline{b_2} * \overline{b_1})$

$= \overline{b_2} * (b \oplus \overline{b_1})$ (axiom B4)

$= \overline{b_2} * 1$

$= \overline{b_2}$ (axiom B1).

We have shown that $\overline{b_1} = \overline{b_2}$ and so we can conclude that the complement is unique.

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4. Find the inverse (if it exists) of the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 5 \end{pmatrix}$$

Sol

$$A = \left\langle \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 5 & 0 & 0 & 1 \end{array} \right\rangle \xrightarrow[r_3 \leftrightarrow r_3 - 4r_{11}]{r_3 \leftrightarrow r_2 - 2r_{11}} \left\langle \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & -2 & 1 & 0 \\ 0 & -3 & 1 & -4 & 0 & 1 \end{array} \right\rangle$$

$$\xrightarrow{r_3 \leftrightarrow \frac{-1}{3} r_2} \left\langle \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} & 0 \\ 0 & -3 & 1 & -4 & 0 & 1 \end{array} \right\rangle \xrightarrow[r_3 \to r_3 + 3r_2]{r_1 \leftrightarrow r_1 - r_2} \left\langle \begin{array}{ccc|ccc} 1 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array} \right\rangle$$

No further sequence of elementary row operations will complete the conversion of the matrix *A* to *I*3. The matrix *A* is not row-equivalent to *I*3. Therefore,

A does not have an inverse and is a **singular matrix**.

Fourth Question:

1. Find the value λ which make the following system has an infinite number of solutions and **write** the solution set of the system:

$$5x+2y-z=1$$
$$2x+3y+4z=7$$
$$4x-5y+\lambda z=\lambda-5$$

Sol

$$(A/b)=\begin{pmatrix}5 & 5 & -1 & 1\\ 2 & 3 & 4 & 7\\ 4 & -5 & \lambda & \lambda-5\end{pmatrix}\xrightarrow[\substack{r_2\to -3r_1+r\\ r_3\to -r_1+r_3}]{r_2\leftrightarrow r_1}\begin{pmatrix}1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5}\\ 0 & \frac{11}{5} & \frac{22}{5} & \frac{33}{5}\\ 0 & \frac{-33}{5} & 4+5\lambda & \lambda-\frac{29}{5}\end{pmatrix}$$

$$\xrightarrow{r}\begin{pmatrix}1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5}\\ 0 & 1 & 2 & 3\\ 0 & 0 & 70+5\lambda & 5\lambda+70\end{pmatrix}$$

the value λ which make the system has an infinite number of solutions is $\lambda=-14$

take $\lambda=-14$ we have that:

$$\begin{pmatrix}1 & \frac{2}{5} & \frac{-1}{5} & \frac{1}{5}\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$y+2z=3$$
$$x+\frac{2}{5}y-\frac{1}{5}z=\frac{1}{5}\Rightarrow 5x+2y-z=1$$

Take $z=s$, so we have $y=3-2s$,

$$x=3s-5$$

Set solution is $\{3s\text{-}5,3\text{-}2s,s\}$

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Faculty of Computers and Informatics | | Subject : Math. 3
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2. Define *Trees, the complete graph* K_n *, the complete bipartite graph* $K_{r,s}$ and *Eulerian path*, and for **which values** of *n, r, s* , the graphs $K_n, K_{r,s}$ are **Eulerian**?

Sol

A trees is a connected graph which contains no cycles.

the complete graph K_n is a simple graph in which every pair of distinct vertices is joined by an edge.

A complete bipartite graph is a bipartite graph such that every vertex of V_1 is joined to every vertex of V_2 by a unique edge.
An Eulerian path in a graph G is a closed path which includes every edge of G . A graph is said to be Eulerian if it has at least one Eulerian path.

The complete graph K_n is (n−1)-regular—every vertex has degree n−1.
Since it is connected, K_n is Eulerian if and only if n is odd (so that n − 1is even).
A complete bipartite graph $K_{r,s}$ is Eulerian if and only if *r*,*s* is even.
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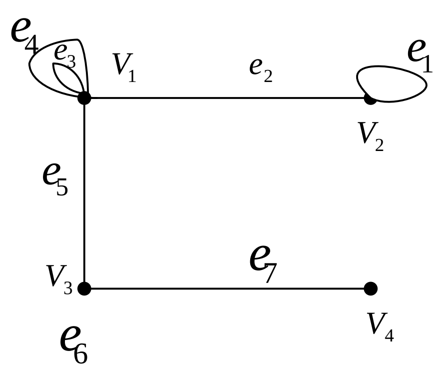
3. Find the matrix A^2 , where A be the adjacency matrix, for the following graph: and **write** all edge sequences of length 2 joining v_1, v_2 .



Sol

$$A=\begin{pmatrix}2&1&1&0\\1&1&0&0\\1&0&0&1\\0&0&1&0\end{pmatrix},\quad A^2=\begin{pmatrix}6&3&2&1\\3&2&1&0\\2&1&2&0\\1&0&0&1\end{pmatrix}$$

e_2e_1; $e_3\, e_2$; $e_4\, e_2$.