Benha University
Faculty of Computers
& Informatics

محملين + تخلفات

Subject: Mathematics(2)
Time: 3 Hours
Class: 1st Year Students,**term2**
(May2016) Final Exam

جامعة بنها – كلية الحاسبات والمعلومات

الفرقة: الاولى- طلاب محملين بالمادة

يوم الامتحان: الاحد

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المادة : رياضيات (٢) - طلاب محملين بالمادة

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مدرس بقسم الرياضيات بكلية العلوم

الامتحان+ نموذج إجابة

ورقة كاملة

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***Answer the following questions*:**

Question1.

a) Find the Maclaurine series for $f(x)=\ln(1+x)$. Use this series to find sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. **[10 marks]**

b) Determine which of the following converge or diverge:

(i) $\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$ (ii) $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\,\ln n}$ (iii) $\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{1+n^2}$ **[15 mark]**

c) Which of the following series converges absolutely, which converges conditionally and which diverge?

(i) $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}\,\ln n}$ (ii) $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n!}$ (iii) $\sum_{n=1}^{\infty}\cos(n\pi)\frac{n+1}{n^2}$ **[15 mark]**

Question2.

Find the solution of each of the following differential equations:

i) $y'=\cot(x+y)-1.$ **ii)** $y'=\frac{y-x}{y-x+2}.$

iii) $y'+y=\frac{x}{y}.$ **iv)** $(x^2+y^2)\,dx+2xy\,dy=0.$ **[35 mark]**

v) $xe^{x^2+y^2}\,dx+y(e^{x^2+y^2}+1)\,dy=0,\quad y(0)=0.$

Question3.

a) Find the following integral $\int\frac{1}{x^2+x-2}\,dx$ **[10 marks]**

b) Determine whether each integral is convergent or divergent. Evaluate those that are convergent

1– $\int_1^{\infty}\frac{1}{x^2}\,dx,$ 2– $\int_{-\infty}^{0}\frac{1}{x+1}\,dx,$ 3– $\int_1^{\infty}\frac{1}{9+x^2}\,dx$

4– $\int_1^{2}\frac{1}{1-x}\,dx,$ 5– $\int_0^{\pi}\tan x\,dx,$ 6– $\int_{-1}^{1}\frac{1}{x}\,dx$ **[30 mark]**

GOOD LUCK,

Dr. Mohamed Nasr – Dr. Essam Awwad

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The answer

Answer Question 1:

a)

$f(x)=\ln(1+x) \qquad f(0)=\ln 1=0$

$f'(x)=\dfrac{1}{1+x} \qquad f'(0)=\dfrac{1}{1}=1$

$f''(x)=\dfrac{-1}{(1+x)^2} \qquad f''(0)=\dfrac{-1}{1}=-1$

$f'''(x)=\dfrac{2}{(1+x)^3} \qquad f'''(0)=\dfrac{2}{1}=2=2!$

$\cdots\cdots\cdots$

$$f(x)=\ln(1+x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

$$=\frac{x}{1!}-\frac{x^2}{2!}+\frac{2x^3}{3!}-\cdots$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

put $x=1$, we get

$$\ln(2)=1-\frac{1}{2}+\frac{1}{3}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

b)

(i) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} (\ln n - \ln(n+1))$

$$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + (\ln n - \ln(n+1))$$

$$S_n = \ln 1 - \ln(n+1) = -\ln(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(-\ln(n+1)) = -\infty$$

$$\Rightarrow \{S_n\} \quad \& \quad \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) \quad \text{diverge.}$$

(ii) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\,\ln n}$

by using limit comparison test and using L'Hopital rule, we have

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\,\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{n}{2\sqrt{n}} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \sqrt{n} = \infty > 0,$$

$$\text{since } \sum \frac{1}{n} \text{ diverge} \quad \Rightarrow \quad \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\,\ln n} \quad \text{diverge.}$$

(iii) $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$

$$I = \int_1^{\infty} \frac{\tan^{-1} x}{1+x^2}\,dx = \lim_{n\to\infty} \int_1^{b} \frac{\tan^{-1} x}{1+x^2}\,dx = \frac{1}{2} \lim_{n\to\infty} \left[\left(\tan^{-1} x\right)^2\right]_1^b$$

$$= \frac{1}{2} \lim_{n\to\infty} \left[\left(\tan^{-1} b\right)^2 - \frac{\pi^2}{16}\right] = \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right] = \frac{3\pi^2}{32} \neq \infty$$

$$I \text{ converge} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2} \quad \text{converge [Integral test].}$$

c)

(i) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}\ \ln n}$

$$a_n = \frac{1}{\sqrt{n}\ \ln n}, \qquad a_{n+1} = \frac{1}{\sqrt{n+1}\ \ln(n+1)} \quad \Rightarrow$$

$$(1) \quad a_n > 0 \qquad \forall n > 1$$

$$(2) \quad a_{n+1} \le a_n \qquad \forall n > 1$$

$$(3) \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}\ \ln n} = 0$$

$$\text{from (1),(2) and (3)}: \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}\ \ln n} \quad \text{Converge,} \qquad (*)$$

$$\text{from b) (ii)}: \sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}\ \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ \ln n} \quad \text{Diverge,} \quad (**)$$

$$\text{from (*) and (**)}: \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}\ \ln n} \quad \text{Converges Conditionally.}$$

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{1}{n!} \quad \text{Converges [Ratio test]} \quad (***)$$

Ratio test :

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = 0 < 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n!} \quad \text{Converges}$$

$$\text{from (***)}: \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \quad \text{Converges absolutely.}$$

(iii) $\sum_{n=1}^{\infty} \cos(n\pi)\frac{n+1}{n^2} = \sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$

$a_n = \frac{n+1}{n^2}$

(1) $a_n > 0 \qquad \forall n \geq 1$

(2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$

(3) $f(x) = \frac{x+1}{x^2}$

$f'(x) = -\left[\frac{x^2+2x}{x^4}\right] \Rightarrow f'(x) < 0 \quad \forall\, x > 0$

$f(x)$ decreasing $\forall\, x > 0$

$a_n = f(n)$ decreasing $\forall\, n > 0$

$\Rightarrow \quad a_{n+1} \leq a_n \qquad \forall\, n \geq 1$

from (1),(2) and (3): $\sum_{n=1}^{\infty} \cos(n\pi)\frac{n+1}{n^2}$ Converges, $\qquad (*)$

$\sum_{n=2}^{\infty} \left|\cos(n\pi)\frac{n+1}{n^2}\right| = \sum_{n=2}^{\infty} \frac{n+1}{n^2}$ Diverges[Comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$], $\qquad (**)$

from $(*)$ and $(**)$: $\sum_{n=1}^{\infty} \cos(n\pi)\frac{n+1}{n^2}$ Converges Conditional

Limit comparison test:

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = 1,$$

since $\sum \frac{1}{n}$ Diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{n+1}{n^2}$ Diverges.

Answer Question 2:

i) $y' = \cot(x+y) - 1$

$$u = x + y \quad \Rightarrow \quad u' = 1 + y' \quad \Rightarrow \quad u' = \cot u = \frac{\cos u}{\sin u}$$

$$\frac{\sin u}{\cos u}\, du = dx \;\Rightarrow\; -\ln \cos u = x - \ln C \;\Rightarrow\; \cos(x+y) = C\, e^{-x}.$$

ii) $y' = \dfrac{y-x}{y-x+2}$

$$u = y - x \Rightarrow \quad u' = y' - 1 \quad \Rightarrow \quad u' = \frac{u}{u+2} - 1 = \frac{-2}{u+2} \quad \Rightarrow$$

$$(u+2)\, du = -2\, dx \;\Rightarrow\; \frac{u^2}{2} + 2u = -2x + \frac{C}{2} \Rightarrow (y-x)^2 + 4y = C.$$

iii) $y' + y = \dfrac{x}{y}, \quad \times 2y, \text{ put } u = y^2 \;\Rightarrow$

$$2yy' + 2y^2 = 2x \Rightarrow \; u' + 2u = 2x$$

$$P = 2, \; \int P\, dx = 2x, \quad \rho = e^{2x},$$

$$Q = 2x, \; \int \rho Q\, dx = \int e^{2x}\, 2x\, dx = xe^{2x} - \frac{1}{2} e^{2x}$$

$$e^{2x} u = (x - \frac{1}{2}) e^{2x} + C \quad \Rightarrow \quad u = x - \frac{1}{2} + Ce^{-2x} = y^2.$$

iv) $(x^2 + y^2)\, dx + 2xy\, dy = 0, \quad y = xv \;\Rightarrow\; \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$(1 + v^2) + 2v(v + x\frac{dv}{dx}) = 0 \;\Rightarrow 1 + 3v^2 + 2v\, x\frac{dv}{dx} = 0 \;\Rightarrow$$

$$\frac{dx}{x} + \frac{2v\, dv}{1 + 3v^2} = 0 \quad \Rightarrow \; \ln x + \frac{1}{3}\ln(1 + 3v^2) = \frac{1}{3}\ln C$$

$$x^3(1 + 3v^2) = C \quad \Rightarrow \quad x(x^2 + 3y^2) = C.$$

v) $xe^{x^2+y^2}\,dx + y(e^{x^2+y^2}+1)\,dy = 0,\quad y(0)=0.$

$$M = xe^{x^2+y^2}\quad,\quad N = y(e^{x^2+y^2}+1)$$

$$M_y = 2xy\,e^{x^2+y^2},\quad N_x = 2xy\,e^{x^2+y^2}$$

$$\therefore\ M_y = N_x$$

$$\int_x M\,dx = \frac{1}{2}e^{x^2+y^2}$$

$$\int_y N\,dy = \frac{1}{2}e^{x^2+y^2} + \frac{1}{2}y^2$$

$$e^{x^2+y^2} + y^2 = C$$

since $y(0)=0 \ \Rightarrow\ C=1 \ \Rightarrow$ Particular solution $\ e^{x^2+y^2}+y^2 = 1.$

Answer Question 3:

$$a)\ \int \frac{1}{x^2+x-2}dx = \int \frac{1}{(x-1)(x+2)}dx = \int\left(\frac{A}{x-1}+\frac{B}{x+2}\right)dx$$

$$= A\ln|x-1| + B\ln|x+2| + c$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + c$$

where c is the integration constant.

b)

$$1-\int_1^{\infty}\frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty}(1-\frac{1}{t}) = 1.\quad (Convergent)$$

$$2-\int_{-\infty}^{0}\frac{1}{x+1}dx = \int_{-\infty}^{-2}\frac{1}{x+1}dx + \int_{-2}^{0}\frac{1}{x+1}dx$$

$$\because \int_{-\infty}^{-2}\frac{1}{x+1}dx = \lim_{t\to-\infty}\int_t^{-2}\frac{1}{x+1}dx = \lim_{t\to-\infty}(\ln 1 - \ln|t+1|) = -\infty \quad (Divergent)$$

$$\therefore \int_{-\infty}^{0}\frac{1}{x+1}dx = \int_{-\infty}^{-2}\frac{1}{x+1}dx + \int_{-2}^{0}\frac{1}{x+1}dx \qquad (Divergent)$$

$$3-\int_1^\infty \frac{1}{9+x^2}\,dx=\lim_{t\to\infty}\int_1^t \frac{1}{9+x^2}\,dx=\lim_{t\to\infty}\left(\frac{1}{3}\left(\tan^{-1}\frac{t}{3}-\tan^{-1}\frac{1}{3}\right)\right)$$

$$=\frac{1}{3}\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{3}\right) \qquad (Convergent)$$

$$4-\int_1^2 \frac{1}{1-x}\,dx=\lim_{t\to1^+}\int_t^2 \frac{1}{1-x}\,dx=\lim_{t\to1^+}(\ln|1-t|-\ln 1)=-\infty. \quad (Divergent)$$

$$5-\int_0^\pi \tan x\,dx=\int_0^{\frac{\pi}{2}} \tan x\,dx + \int_{\frac{\pi}{2}}^{\pi} \tan x\,dx$$

$$\because \int_0^{\frac{\pi}{2}} \tan x\,dx =\lim_{t\to\frac{\pi}{2}^-}\int_0^t \tan x\,dx=\lim_{t\to\frac{\pi}{2}^-}\left(\ln\frac{1}{\cos t}-1\right)=\infty \quad (Divergent)$$

$$\therefore \int_0^\pi \tan x\,dx=\int_0^{\frac{\pi}{2}} \tan x\,dx + \int_{\frac{\pi}{2}}^{\pi} \tan x\,dx=\infty. \qquad (Divergent)$$

$$6-\int_{-1}^1 \frac{1}{x}\,dx=\int_{-1}^0 \frac{1}{x}\,dx+\int_0^1 \frac{1}{x}\,dx$$

$$\therefore \int_0^1 \frac{1}{x}\,dx=\lim_{t\to0^+}\int_t^1 \frac{1}{x}\,dx=\lim_{t\to0^+}(1-\ln|t|)=\infty \qquad (Divergent)$$

$$\therefore \int_{-1}^1 \frac{1}{x}\,dx=\int_{-1}^0 \frac{1}{x}\,dx+\int_0^1 \frac{1}{x}\,dx=\infty \qquad (Divergent)$$